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Analysis of the Meanings of the Antiderivative Used by Students of the First Engineering Courses

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Abstract In this article, we present the results of the administration of a questionnaire designed to evaluate the understanding that civil engineering students have of the *antiderivative*. The questionnaire was simultaneously administered to samples of Mexican and Colombian students. For the analysis of the answers, we used some theoretical and methodological notions provided by the theoretical model known as Onto-Semiotic Approach (OSA) to mathematical cognition and instruction. The results revealed the meanings of the antiderivative that are more predominantly used by civil

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engineering students. Also, the comparison between the mathematical activity of Mexican and Colombian students provides information that allows concluding that the meanings mobilized could be shared among their communities and are not particular of their classroom or university.

Keywords Antiderivative · Calculus · Engineering students · Understanding

Background

Mathematics education of engineering students is a topic that, for some years, has been of growing concern among researchers in the field of Mathematics Education as well as university mathematics teachers and professional associations (Bingolbali, Monaghan, & Roper, 2007). In this regard, there have been several studies that have dealt with the issue of how to address different mathematical notions in engineering contexts (Gnedenko & Khalil, 1979; Hieb, Lyle, Ralston, & Chariker, 2015; Neubert, Khavanin, Worley, & Kaabouch, 2014; Randahl, 2012; Sonnert & Sadler, 2014). The suggestions given by these studies focused on the type of problems used to introduce mathematical notions (real-life problems, modelling of real situations in engineering contexts, etc.), the impact of technological resources and textbooks for the teaching of mathematics to engineers, and even motivational factors. As shown in Artigue, Batanero, and Kent (2007), other works evoke the study of the differences in the way of thinking mathematics between mathematics and engineering students (Bingolbali et al., 2007; Jones, 2015; Maull & Berry, 2000).

This article aims at identifying and characterizing the meanings of the antiderivative that engineering students, specifically students from Civil Engineering Programs, use in their mathematical practices in connection to certain tasks assigned to them on this mathematical notion. For this purpose, we administered a questionnaire to two groups of civil engineering students, one from a Mexican University and another from a Colombian University. The analysis of the answers to the questionnaire shows which are the meanings that future civil engineers give to the antiderivative and how these relate to the partial meaning that make up the holistic meaning of this mathematical notion (Pino-Fan, Godino, & Font, 2016).

Regarding the *antiderivative* notion, few studies have been conducted in the context of other programs different from mathematics (e.g. Jones, 2015), and the treatment that is frequently given to this mathematical object is from the point of view of the integral (Contreras, Ordóñez & Wilhelmi, 2010; Crisóstomo, 2012; Jones, 2013). However, the importance of this mathematical object has been acknowledged and has been given more prominence and identity in research in Mathematics Education (e.g. Hall, 2010; Sealey, 2006; Sealey, 2014). Thus, research conducted on the antiderivative has focused on the way students reflect on the rules of integration (Metaxas, 2007; Posso, Uzuriaga & Martínez, 2011), its historical-epistemological meanings (Gordillo & Pino-Fan, 2016), and the use of technology for its introduction and study (Ponce-Campuzano & Rivera-Figueroa, 2011), including an approach from the perspective of the theory of objectification that involves a continuous development of the meanings according to the elements used in class (Kouropatov & Dreyfus, 2014; Swidan & Yerushalmy, 2014).

But, why is a study on the different meanings of the fundamental notions of calculus relevant? More specifically, why is it important to study the meanings of the antiderivative taught and learnt in engineering programs? To answer these questions, it is advisable to do a brief history review. At the beginning of the XX century, the advantages, from the point of view of rigour, present in the arithmetical version of calculus of Cauchy-Weierstrass, caused mathematicians to prescind from any other version of calculus, and particularly, any allusion to evanescent or infinitely small quantities. From then on, infinitesimal notions were no longer used in calculus textbooks. As a consequence, courses of calculus were organized with the notion of limit as a central concept (Arcos & Sepúlveda, 2014). In non-standard analysis, the original method of reasoning by means of infinitesimals is validated and vindicated. According to Arcos (2004), the method of limits is laborious, not very intuitive and far from reality, but it is by far the most commonly used nowadays. For this reason, this author offers didactical suggestions based on the non-standard analysis for the study of calculus because he considers that reasoning by means of infinitesimals is very appropriate for engineering students. According to several authors (e.g. Arcos & Sepúlveda, 2014), the mathematics of infinitesimals can be better linked to modelling and represents an introduction to calculus that reacts better to the mathematics that engineers have to use in their profession.

The different meanings that the fundamental notions of calculus have had throughout history show the complexity of these mathematical objects. For example, and in relation to the formal notion of limit, Jones (2013) points out: 'While the limit is fundamental to calculus, the derivative and the integral have additional layers of meaning above and beyond the limit, as well as meanings that do not necessarily require accessing the concept of a limit' (p. 122).

Such complexity involves a relevant problem: What are the different meanings of the fundamental notions of calculus (such as the antiderivative) that engineers should be taught? Which of these meanings have future engineers learnt? Our study addresses a partial aspect of this discussion, by inquiring into the features of the personal meanings of the antiderivative that engineering students have, after a process of conventional teaching.

In the next section, we present the theoretical notions and the methodology that we used to conduct our study. Then, in the third section, we conduct the analysis of the answers given by the Mexican and Colombian students and discuss about the base of comparison of the results. Finally, in the section of final reflections, we comment about the main contributions and limitations of our study, and identify additional lines of investigation.

Theoretical Framework

In order to conduct this study, we considered the theoretical model known as the Onto-Semiotic Approach (OSA) to mathematical cognition and instruction (Godino, Batanero, & Font, 2007). In OSA, the notion of *mathematical practices* plays and important role in the teaching and learning of mathematics. OSA assumes certain pragmatism when considering mathematical objects as entities that emerge from the practices conducted in a field of problems (Godino & Batanero, 1994). Font, Godino and Gallardo (2013) points it out this way, 'Our ontological proposal is derived from mathematical practice, this being the basic context in which individuals gain their experience and in which mathematical objects emerge. Consequently, the object here acquires a status derived from the practice that precedes it.' (p. 104). In this sense, in OSA, the meaning of mathematical objects is conceived from a pragmatic-anthropological perspective which considers the relativity of the context in which these are used. In other words, the meaning of a mathematical object can be defined as the system of operative and discursive practices that a person (or an institution) develops in order to solve certain type of situations problems in which such object intervenes (Godino & Batanero, 1994). Thus, the meaning of a mathematical object can also be considered from two perspectives, institutional and personal.

In order to conduct a 'finer' and more systematic analysis of the mathematical practices developed regarding certain problems, OSA introduces a typology of primary mathematical entities (or primary mathematical objects) that intervene in the systems of practices: problems, linguistic elements, definitions, propositions, procedures and arguments. These primary mathematical objects are related among themselves forming nets of intervening objects that emerge from the systems of practices, which in OSA are known as *configurations*. These configurations can be epistemic (institutional nets of primary mathematical objects) or cognitive (personal nets of primary mathematical objects). We use the notion of *cognitive configuration* to analyse the mathematical practices performed by civil engineering students regarding the solutions to the tasks of the questionnaire. Within OSA, such notion—cognitive configuration—is essential for the study of understanding because the pragmatic positioning of OSA leads to consider understanding, basically, as a competence and not as a mental process; in other words, it is considered that a subject understands a certain mathematical object when he/she uses it in a competent manner in different practices. In this way, understanding, as stated by Pino-Fan (2014), has to do with the relations—seen from the perspective of mathematical congruence—that have to be established among all the elements that intervene in the cognitive configuration that the subject activates to solve certain situations/problems.

That said, in the framework of OSA and in relation to the complexity of the object integral, several works have been developed, which somehow, are related to the antiderivative. For example, Contreras, Ordóñez and Wilhelmi (2010) consider the following epistemic configurations: (1) geometric, (2) the result of a process of change, (3) inverse of the derivative, (4) approximation to limit, (5) generalized (Lebesgue, Riemann, etc.), (6) algebraic and (7) numerical methods. On the other hand, Crisóstomo (2012), in his doctoral dissertation considers—based on the net of epistemic configurations suggested by Contreras et al. (2010)—useful to differentiate eight different types of configurations, which he calls Intuitive, Primitive, Geometric, Summational, Approximated, Extra mathematical, Accumulated and Technological, placing the Fundamental Theorem of Calculus (FTC) as a primary object that is central to the epistemic configuration called primitive, although it also appears in the geometric, summational, extra mathematical and technological configurations.

It should be stressed that the study carried out in previous works about the antiderivative is indirect and always from the point of view of the integral. For example, the contributions of Sealey (2006, p. 46) are not taken into consideration, when he points out:

(…) many real-world applications involve functions that do not have an antiderivative that can be expressed in terms of elementary functions. For example, the antiderivative of the function $f(x)=e^{x^2}$ cannot be expressed in terms of elementary functions. Thus, the Fundamental Theorem of Calculus could not be applied, and other methods for evaluating the definite integral, such as Riemann sums would be needed.

What was stated above would once again lead us to consider the other meanings (or layers of meanings) mentioned by Jones (2013), and that, in the case of the antiderivative, we have characterized with the study of Gordillo and Pino-Fan (2016). In this work, the main role is given to the antiderivative, which is granted identity as a research object, differentiating it from the notion of integral (Wagner, 2015), taking into account the diverse partial meanings of reference of the antiderivative for the creation of the questionnaire.

Methodological Aspects

This study uses, mainly, a qualitative methodology (Cohen, Manion, & Morrison, 2011), since it is an exploratory study that considers the observation of qualitative variables (the type of cognitive configuration connected to the practices on antiderivative). In addition, descriptive statistics are used when analysing percentages for quantitative variables (answers' degree of accuracy: correct answers, partially correct answers and incorrect answers).

For the study of the qualitative variable, the students' protocols of answers were analysed, and the primary mathematical objects that intervened in the cognitive configurations related to their mathematical practices were described in a systematic way.

For data gathering purposes, a questionnaire was administered (and will be described next), which was specifically designed to evaluate university students' understanding of the antiderivative. The questionnaire was administered in one session of 2 h, in different days in each of the two participant universities. Prior to these sessions, and in order to motivate students, they were informed that they would be part of a research study, and their anonymity was guaranteed by giving the possibility to write 'Subject-Male' or 'Subject-Female' to those who did not want to put their names on the test.

The Questionnaire

The questionnaire that we used to gather data was designed to evaluate students' understanding on antiderivative and is composed of 11 tasks (Gordillo, Pino-Fan, Font, & Ponce-Campuzano, 2015). Each of these tasks is closely related to one of the four partial meanings of the antiderivative that were identified through a historical-epistemological study that aimed at reconstructing the 'holistic meaning of reference' for such mathematical object (Gordillo & Pino-Fan, 2016). Figure 1 shows a summary of the characteristics and goals pursued by each of the tasks. A complete analysis of the content evaluated in each task, as well as the identification of the possible difficulties that students may encounter to solve them, can be found in Gordillo et al. (2015).

Subjects and Context

The questionnaire was administered to an intentional sample of Civil Engineering students, specifically, two groups. The first group was composed of 23 students of

Task	Expected mathematical practice	Representation activated	Partial meaning activated
Task 1: Meanings of the antiderivative	Description of the personal meanings and definitions for the antiderivative	Verbal/Written	Global
Task 2: Structured Synoptic model	Relation of the antiderivative to other mathematical objects of calculus	Concept Map/Graph	Global
Calculation of the primitive function (parts A and B)	Construction of a family of functions from a derivative function	Symbolic, graphic and tabular	Differential-sum
Task 4: Graphic exploration of the antiderivative	Treatment of the graphic representation of the antiderivative	Graphic	Tangent- squaring
Task 5: Difference integral-antiderivative	Description of the conceptual differences between the notions of integral and antiderivative.	Verbal, Written and symbolic	Elementary functions
Task 6: Elementary functions	Identification of the derivative function as elementary function	Verbal, Written and symbolic	Elementary functions
Task 7: Rules of 'antiderivatives'	Identification of the antiderivative from a basic rule of derivation	Symbolic	Tangents- squaring
Task 8: Notations of a derivative function	Identification of a way to denote a derivative function	Symbolic	Tangents- squaring
Task 9: Applications of the antiderivative in Economy	Application of the mathematical object antiderivative in Economics sciences	Verbal, Written and symbolic	Tangents- squaring
Task 10: Solving of ordinary differential equations	Use of the antiderivative for solving differential equations	Verbal, Written and symbolic	Fluents- Fluxions
Task 11: Applications of the antiderivative in Physics	Application of the mathematical object antiderivative in context of physics	Verbal, Written and symbolic	Fluents- fluxions

Fig. 1 Summary of the characteristics of the tasks of the questionnaire

the Civil Engineering Program of the Faculty of Environment and Natural Resources of the Universidad Distrital Francisco José de Caldas in Colombia. The second group was composed of 23 students of the Civil Engineering Program of the Faculty of Engineering of the Universidad Autónoma de Querétaro in Mexico. An essential requisite for the selection of the students was that, at the moment of responding to the questionnaire, they had taken courses of Integral Calculus. This aspect was fulfilled by the 46 students who took the test.

It should be noted that the role of the authors was limited to the administration of the questionnaire, the authors had never taught these students, nor did they have any link with the professor who held the courses at the time of the administration of the questionnaire.

Analysis of Data

In this section, we present the analysis of the answers given by the students of the two groups, Mexican and Colombian. For the analysis of the quantitative variable, we assigned the labels 2, 1, and 0, depending on whether the answers were correct, partially correct or incorrect, respectively. Thus, the highest score that a student could obtain by answering all the questions correctly was 24 points. The first study that we conducted with the variable level of accuracy was done in order to determine if there were significant differences between the Colombian group (*Group 1*) and the Mexican group (*Group 2*). For that purpose, we used the statistical package IBM SPSS (version 22) to conduct a comparison between independent samples. The results of this comparison are presented in Table 1.

As shown in Table 1, in order to verify if there were significant statistical differences between the groups under study, a parametric hypothesis test of analysis of variance was used (ANOVA), which allowed us to compare average scores of the two samples. By means of this ANOVA test to compare the averages, we found that, with a degree of confidence of 95%, there were no significant differences between the average scores of the two groups of engineering students.

Therefore, since there were no statistically significant differences found between the two groups, for the quantitative analysis of the results (next section), the 46 students will be considered as one single sample.

For the analysis of the qualitative variable, we used the notion of *cognitive configuration*, which allowed us to describe in a systematic way the primary mathematical objects (linguistic elements, definitions, propositions, procedures and arguments) that form the mathematical practices of the students, in connection to the tasks of the questionnaire.

Analysis of the Answers of the Mexican and Colombian Engineering Students

In this section, we present the results of the quantitative and qualitative analysis of each of the tasks of the questionnaire.

Task 1: Meanings of the Antiderivative

Given the nature of this first task, only correct answers (answers in which at least one of the partial meanings of the antiderivative was expressed in verbal/written form) and incorrect answers (answers in which any of the partial meanings of the antiderivative were enunciated) were considered. The students did not have difficulties solving the task, answering 82.6% correctly. Table 2 shows a summary of answers provided by the students.

As shown in the table above, a high percentage of Mexican students (13) as well as Colombian (11) answered that the antiderivative is 'the inverse process of derivation'. This first general approach to the conceptions that students have of the antiderivative shows that more than half of them (52.2%) think of the antiderivative as a *procedure* (operation) that allows to find the 'original function' from which certain derived function comes from. Out of the 46 students, only one student from Mexico answered

Table 1 Statistical summary for the total scoring by groups

Group	*N*	Average	Standard deviation	Standard error	Confidence interval for the average at 95%		Minimum	Maximum
					Lower limit	Upper limit		
1	23	14.087	3.1754	0.6621	12.714	15.460	9.0	21.0
2	23	14.000	3.3166	0.6916	12.566	15.434	7.0	20.0
Total	46	14.043	3.2108	0.4734	13.090	14.997	7.0	21.0

Table 2 Frequencies and percentages for the type of cognitive configuration activated in task 1

Types of cognitive configuration	Frequency by group		Total frequency	%
	1	2		
Family of functions	0	1	1	2.2
'Inverse process' of derivation	11	13	24	52.2
Primitive of a function	7	1	8	17.4
Indefinite integral	0	5	5	10.8
Absence of meaning	5	3	8	17.2
Total	23	23	46	100

that the antiderivative is a 'family of functions'. The solutions that we have labelled as 'absence of meaning', which refer to incorrect answers from the point of view of the level of accuracy, are answers in which the students did not give any meaning to the antiderivative, providing answers of the type 'the antiderivative is the area below the curve', 'the antiderivative is obtained from the fundamental theorem of calculus', 'the antiderivative is a function f of $f=f'$', 'the antiderivative is a mathematical form through which some real life problems can be solved'.

Task 2: Structured Synoptic Model

Regarding the goals that we pursued with these tasks, analysing relations, links and connections that students establish between the antiderivative and other mathematical objects only correct and partially correct answers were considered for this task. Table 3 shows the results regarding the level of accuracy of the answers to task 2.

'Partially correct' answers were those in which the students established a relation between at least five of the linguistic elements provided (indefinite integral, $\frac{dy}{dx}$, velocity, derivative, integral, area between two curves, $f'(x)$, antiderivative, $\int_a^b f(x)dx$, definite integral, fundamental theorem of calculus). An answer was considered as 'correct' if a relation was established among at least ten of the expressions provided. Regarding the type of relations established, it was possible to classify them into three types (Table 4).

The type of 'synoptic-basic' answers refers to those answers in which only few of the linguistic elements provided were related, and the sense or direction of the connection was not provided nor justifications given for them. The label 'synoptic-intermediate' refers to answers in which at least ten of the expressions given were related and the sense and direction of the connection, and justifications for such connections were provided; however, no differences were found between the integral and the antiderivative. On the other hand, the answers that we have categorized as 'synoptic-advanced', apart from establishing connections in a similar way as in the

Table 3 Frequencies and percentages for the level of accuracy for task 2

Level of accuracy	Total frequency	%
Correct	13	28.3
Partially correct	33	71.7
Total	46	100

Table 4 Frequencies and percentages for the type of cognitive configuration activated in task 2

Types of cognitive configuration	Frequency by group		Total frequency	%
	1	2		
Synoptic—Basic	13	18	31	67.4
Synoptic—Intermediate	7	2	9	19.6
Synoptic—Advanced	3	3	6	13.0
Total	23	23	46	100

answers of the previous category, differences between the notions of integral and antiderivative are also found and justified. The answers given to this task once again provide evidence of the conception that most students have about the antiderivative as the inverse process of derivation, and the equivalence that they establish between the indefinite integral and the antiderivative.

Task 3: Calculation of the Primitive Function

Task 3 was composed of two parts. For the first part, part A, we considered as correct all the answers in which a valid symbolic expression was provided for $f(x)$, while incorrect answers were all the answers that did not provide valid symbolic expressions for $f(x)$. For part B, all the answers which provided a second expression for $f(x)$, different from the one given in part A and with valid justifications, were considered as correct. All the answers in which it was explicitly or implicitly mentioned that it was not possible to find a second expression for $f(x)$ were considered as incorrect.

The students did not have problems to provide a symbolic expression for $f(x)$ in part A of the task, with 87% (40) of them giving a correct answer. However, similar to what happened in the study of Pino-Fan (2014), the students had more difficulties answering part B of the task, with 50% (23) of them giving a second valid expression for $f(x)$ different to the one provided in part A. Tables 5 and 6 show the frequencies and percentages of the types of cognitive configuration activated in the answers to parts A and B of the task.

We could identify two types of cognitive configurations from the answers provided by the students to part A of the task. The first type 'graphic-technical' refers to the answers in which, from the data given in the table, a graphic representation is provided from which the algebraic expression is obtained (graphic and symbolic linguistic elements, respectively) for the derivative. Subsequently, an expression for $f(x)$ is found

Table 5 Frequencies and percentages for the type of cognitive configuration activated in task 3-A

Types of cognitive configuration	Frequency by group		Total frequency	%
	1	2		
Graphic-technical	1	1	2	4.4
Numeric-technical	17	21	38	82.6
There is no solution	5	1	6	13.0
Total	23	23	46	100

Table 6 Frequencies and percentages for the type of cognitive configuration activated in task 3-B

Types of cognitive configuration	Frequency by group		Total frequency	%
	1	2		
Wrong interpretation about the uniqueness of the derivative	3	3	6	13.0
Equivalent functions	4	0	4	8.7
Advanced	12	11	23	50.0
No solution provided	4	9	13	28.3
Total	23	23	46	100

from the argumentations and procedures centred on the 'rules' (propositions) of derivation. The second type of cognitive configuration, 'numeric-technical', refers to the answers in which a pattern (propositions) that allows establishing the rule of correspondence that defines the derivative (definition) is determined from the data provided in the table. Later, from the argumentations and procedures centred on the rules of derivation, an expression for $f(x)$ is found.

Regarding the cognitive configurations connected to the answers to part B of the task, we found three types. The first type, 'wrong interpretation of the uniqueness of the derivative', are answers in which the students show a wrong conception about the uniqueness of the derivative at a point and the derived function, providing answers of the type 'it is not possible to find another expression for $f(x)$ because for $f'(x)$ there is one and only one $f(x)$, and vice versa'. The second type of configuration, 'equivalent functions', is related to the answers in which, explicitly or implicitly, by means of the use of equivalent functions (concept/definition), some algebraic operations are developed (procedures that serve as arguments) to show that it is not possible to find another different function. The third type of cognitive configuration, 'advanced solution', was activated in answers in which the procedures and their justifications explicitly establish a connection among concepts such as antiderivative, the fundamental theorem of calculus, rules of integration, etc., to point out with the proposition 'another expression for $f(x)$ can be any member of the family of functions $f(x) = x^2 + C$', that it is, indeed, possible to find another expression for $f(x)$. As we can observe, 50% of the students (12 Colombian and 11 Mexican) mobilized the third type of configuration to provide their answers. Regarding the antiderivative, the third type of configuration brings associated the meaning of inverse process of derivation.

Task 4: Graphic Exploration of the Antiderivative

For this task, we only considered correct answers (in which the elements that belong to the family of the antiderivative were correctly identified and the way of finding them was justified) and incorrect answers (in which the graph provided did not correspond with the elements of the family of antiderivative for the function provided graphically). Task 4 represented a higher level of difficulty for the students, with only 41.3% (19) answering correctly. Among the mathematical practices that the students performed as part of their answers, we could identify three types of cognitive configurations. Table 7

Table 7 Frequencies and percentages for the type of cognitive configuration activated in task 4

Types of cognitive configuration	Frequency by group		Total frequency	%
	1	2		
Tabular interpretation of the graph	5	9	14	30.4
Particular function	10	6	16	34.8
Advanced	8	3	11	23.9
No solution provided	0	5	5	10.9
Total	23	23	46	100

shows a summary of the results for the type of cognitive configuration activated in the answers to task 4.

As shown in the table above, out of the three configurations identified, the most used by the students was the 'particular function' (34.8%), in which a symbolic expression for the function is obtained from the graph of the function, and through algebraic procedures, it is possible to identify (or try to identify) which are the graphs of the elements of the family of antiderivatives. The second more frequently used type of configuration was the 'tabular interpretation of the graph' (30.4%), which refers to the answers in which a table of values that describe the function given originally is constructed from the graph of the function provided; from the table constructed (and the relations and properties that are established with it), it is possible to try to identify the elements that belong to the family of antiderivatives. The configuration that we have identified as 'advanced' was activated in answers which were characterized by the use of procedures and justifications centred on the properties/propositions of derivation, specifically the criterion for the analysis of the characteristics and construction of graphs of functions, in order to identify graphically the member that belongs to the family of antiderivatives of the function provided.

Task 5: Difference between Integral and Antiderivative

Task 5 aimed at exploring whether the students conceived the integral and the antiderivative as different notions or not. Table 8 shows the results for level of accuracy.

The correct answers were those in which the students pointed out and justified which were the differences between both notions. Partially correct answers were those in which the students mentioned that there were differences, but the differences were not pointed out, or no justification was given, or the justification was not valid (from the institutional point of view). Only 26.1% of the students pointed out that the antiderivative and the integral were the same notion and that the terms were synonyms (Borasi,

Table 8 Frequencies and percentages for the level of accuracy for task 5

Level of accuracy	Total frequency	%
Correct	12	26.1
Partially correct	22	47.8
Incorrect	12	26.1
Total	46	100

Table 9 Frequencies and percentages for the type of configuration activated in the answers to task 5

Types of configuration	Frequency by group		Total frequency	%
	1	2		
Particular-general	4	3	7	15.2
Definitions for the notions	17	14	31	67.4
Examples of use	2	6	8	17.4
Total	23	23	46	100

1992; Hall, 2010). Table 9 shows the justifications provided by the students regarding the difference between these two notions.

As shown above, the most activated cognitive configuration in the answers was 'definitions for the notions', used by 67.4% of the students. Such configuration was activated in answers in which there were arguments regarding the difference between the concepts of antiderivative and integral, providing definitions (personal or institutional) for both notions. For example, '…are different because the integral is a number, while the antiderivative is another function'. The configuration 'examples of use' was the second most activated configuration (two Colombian and six Mexican) and was activated in answers in which there were arguments regarding the difference between both notions by means of concrete examples (problems) of their use or application, for example, 'the integral serves to calculate the area below the curve while the antiderivative serves to obtain a function'. It is important to point out that the examples of use that were provided in this second configuration made reference to the notions involved as process (or procedure) and not from a conceptual point of view. The third type of configuration activated was 'particular-general' (four Colombian and three Mexican), in answers in which the arguments were oriented towards the distinction of the antiderivative as a general case of the definite integral; in other words, the antiderivative was seen as indefinite integral, which is similar to what was found by Borasi (1992).

Task 6: Elementary Functions

Task 6 aimed at mobilizing the meaning of the antiderivative as elementary function. From this perspective, it is common to find in some books of calculus that a function that has antiderivative can be expressed as an elementary function; in other words, it can be expressed as an addition, a subtraction, a multiplication, a division or a composition of other functions using a finite number of algebraic operations. Obviously, there are functions that cannot be expressed as elementary functions, for example, the function $f(x) = e^{x^2}$; therefore, with the expression $\int e^{x2} dx$, it is not possible to find the antiderivative, but it is possible to calculate the integral of the function with definite limits (e.g. $\int_1^5 e^{x^2} dx$, can be calculated by numerical integration). In this context, Table 10 shows the results for the level of accuracy for task 6. The correct answers are those in which it was pointed out that it was possible to find a function in $\mathbb{R}$ that can be integrated but does not have an antiderivative, and valid justifications for the solution were provided. Partially correct answers are those in which it was pointed out that it is possible to find a function with the characteristics mentioned before, but no arguments were provided or the arguments were not at all valid (from an institutional

Table 10 Frequencies and percentages for the level of accuracy for task 6

Level of accuracy	Total frequency	%
Correct	7	15.2
Partially Correct	14	30.4
Incorrect	20	43.5
No answer	5	10.9
Total	46	100

point of view). The answers, in which the student said it was not possible to find a function as such, were considered as incorrect.

As shown above, a high percentage of students, 54.4% (No answer and Incorrect), had difficulties solving the task. Only 15.2% (7) were able to answer correctly, thus mobilizing the intended meaning of the antiderivative. Table 11 shows a summary of the types of cognitive configurations activated in the answers of the students.

Regarding correct answers (Table 10), we find the cognitive configuration 'classic example' that makes reference to the answers in which it was stated that it was possible to find a function with the characteristics required, and the arguments centred on explaining the 'classic' example (problem) that is frequently cited in textbooks, $f(x) = e^{x^2}$. Only seven Colombian students mobilized this configuration. The configuration 'Contradictory particular examples' was activated in answers that mentioned that it was possible to find a function with the characteristics required, and there is an example of a concrete function that does not fulfil the characteristics; in other words, the examples provided are concrete functions that do have an antiderivative and can be integrated. The configuration 'false conception of equality' was mobilized by 19.6% of the students (four Colombian and five Mexican), and it was activated in answers in which the main argument was the proposition 'it is not possible to find a function like that because the notions of antiderivative and integral are the same'.

Almost half of the students, 47.8%, mobilized the configuration 'invalid verbal descriptions', which is connected to answers in which verbal arguments of a general nature are provided, and in which there is an attempt to articulate several mathematical concepts/definitions and properties/propositions such as continuity, derivability and complex functions. However, such articulation is either not sufficient to validly justify the solution to the task or is incongruent from a mathematical point of view.

Table 11 Frequencies and percentages for the type of configuration activated in the answers to task 6

Types of configuration	Frequency by group		Total frequency	%
	1	2		
Classic example	7	0	7	15.2
Contradictory particular examples	2	1	3	6.5
False conception of equality	4	5	9	19.6
Invalid verbal descriptions	9	13	22	47.8
No solution provided	1	4	5	10.9
Total	23	23	46	100

Task 7: Rules of the 'Antiderivative'

In task 7, it was expected that the students, from the symbolic expression $h(x) = f'(x)g(x) + f(x)g'(x)$, were able to identify the proposition of derivation of the product, in other words, that $h(x)$ is the derived function of a function $p(x) = f(x)g(x)$, and therefore, determine that the antiderivative of $h(x)$ is $H(x) = f(x)g(x) + C$, with C belonging to the real numbers. Table 12 shows the results for the level of accuracy of the answers provided.

Even though, the initial hypothesis of the researchers authors of this document was that the students would identify the property of derivation soon, this task ended up being very difficult for them, with 69.5% of students answering incorrectly (or not answering at all). Out of the rest of the students, 13 were able to identify the property of derivation of the product of functions, and only one was able to provide a correct answer to the task, by identifying that the antiderivative was $H(x) = f(x)g(x) + C$. Regarding the cognitive configurations activated in their solutions, we could identify two types, as shown in Table 13.

We can observe in the table above that 50% of the students used the configuration 'algebraic manipulation', which consisted on the algebraic manipulation (procedure) of the symbolic linguistic element $h(x)$, by applying the properties/proposition of integration and derivation, in order to determine the antiderivative of $h(x)$. The second type of cognitive configuration activated was the 'identification of the rule of derivation', activated in 13 answers (28.3%) and it consisted on, as indicated by its name, the identification of the property/proposition 'if $p(x) = f(x)g(x)$, then $p'(x) = h(x) = f'(x)g(x) + f(x)g'(x)$'.

However, out of the 13 students who activated this second type of configuration, only one answered that the antiderivative was $H(x) = f(x)g(x) + C$, while the other 12 students said that the antiderivative was $H(x) = f(x)g(x)$, omitting the constant C. This shows that at least these 12 students do not think of the antiderivative from a conceptual point of view (Kiat, 2005), but on the contrary, they think of it as an inverse process of the integration and more concretely, as a procedure that allows them to obtain the 'result of the inverse operation' directly, just like in multiplying and dividing.

Task 8: Notations of a Derivative Function

Task 8 was intended to make students recognize the operators $\frac{d}{dx}$ and $\int$ as inverse operators that refer to inverse processes. But, it was also intended to make student recognize that the antiderivative, as inverse process, should really be seen as a 'not-so-direct inverse process'. Table 14 shows the results for the level of accuracy of the answers given this task.

Table 12 Frequencies and percentages for the level of accuracy for task 7

Level of accuracy	Total frequency	%
Correct	1	2.2
Partially correct	13	28.3
Incorrect	22	47.8
No answer	10	21.7
Total	46	100

Table 13 Frequencies and percentages for the type of configuration activated in the answers to task 7

Types of configuration	Frequency by group		Total frequency	%
	1	2		
Algebraic manipulation	14	9	23	50.0
Identification of the rule of derivation	4	9	13	28.3
No solution provided	5	5	10	21.7
Total	23	23	46	100

As in the case of task 7, only two students mobilized the knowledge that we were expecting them to mobilize with this task, by correctly providing the antiderivative. A total of 45.7% of students omitted the constant C in their answers, pointing out that the antiderivative was $F(x) = \frac{1}{\sqrt{x^2+1}}$. The other 50% of the students had some difficulties solving the task assigned.

Regarding the type of cognitive configuration activated in their mathematical practices, we could identify two, which are shown in Table 15. The first type of configuration, 'algebraic manipulation', used by 41.3% of the students, is connected to answers in which $f(x)$ was algebraically manipulated to obtain, by means of properties/propositions, a symbolic expression for $f'(x)$, and then, 'integrate' $f'(x)$. Only one out of the 19 students who mobilized this type of configuration in their answers could correctly find that the antiderivative of $f(x)$ was $F(x) = \frac{1}{\sqrt{x^2+1}} + C$.

Furthermore, the configuration 'identification of inverse operators' was activated in answers in which the procedure of calculating the antiderivative of $f(x)$ centred on the identification of $\frac{d}{dx}$ and $\int$ as inverse operators; in other words, the antiderivative as an inverse process of the derivative. However, only one of the 17 students that used this type of configuration came up with the correct answers. The other students gave partially correct answers in which they omitted the constant C, which makes us think once again that these 16 students conceive the antiderivative as a 'direct inverse procedure' which can be used to obtain the same function of origin of the provided derived function.

Task 9: Application of the Antiderivative in Economics

Task 9 was related to an application of the antiderivative in the context of Economics; in other words, given a certain marginal cost function, it was requested to find the total cost function. It is important to note that in such total cost function, the constant k

Table 14 Frequencies and percentages for the level of accuracy for task 8

Level of accuracy	Total frequency	%
Correct	2	4.3
Partially correct	21	45.7
Incorrect	13	28.3
No answer	10	21.7
Total	46	100

Table 15 Frequencies and percentages for the type of configuration activated in the answers to task 8

Types of configuration	Frequency by group		Total frequency	%
	1	2		
Algebraic manipulation of *f*(*x*)	9	10	19	41.3
Identification of inverse operators	6	11	17	37.0
No solution provided	8	2	10	21.7
Total	23	23	46	100

represents the fixed cost of production of a certain amount of a particular product. Table 16 shows the results for the variable level of accuracy.

We must mention that the reason why we labelled some answers as partially correct (17) was that the constant *k* was omitted in the final answer. However, we did not have enough evidence to corroborate why such constant was omitted, so we considered at least the following two hypothesis: (1) The students calculated the antiderivative of the function provided because they identified that the questionnaire was about antiderivatives, which, due to the results obtained in previous tasks about the conceptions that the students have of the antiderivative, would explain the omission; (2) the students did not give meaning to *k* in the economic context. We formulated this second hypothesis because 37% of the students (correct answers), apart from providing the antiderivative correctly, also described the meaning of *k* (fixed cost) and of other terms of the antiderivative (variable costs) correctly, in the context of economics. Table 17 shows the cognitive configurations activated.

In the table above, we can observe three types of cognitive configurations that we could identify in the answers given by the students. These types correspond to the correct, partially correct and incorrect answers, respectively (Table 16). The first type, 'calculation of variable costs', was activated in answers in which the procedure was based on the propositions of 'integration' to come up with an answer of the type '$c(q) = \frac{5q^3}{3} - \frac{5q^3}{3}$'. However, we do not have enough information that can help us determine why the students omitted the constant *k* (that would represent fixed cost), so we only count on the hypothesis previously enounced, of what could have happened. On the other hand, the configuration 'calculation of total cost' is related to the correct answers in which the procedures were centred on the application of propositions of 'integration', to conclude that the total cost function was defined as $c(q) = \frac{5q^3}{3} - \frac{5q^3}{3} + k$, and in the arguments, it was correctly identified that *k* represented fixed costs, while $\frac{5q^3}{3} - \frac{5q^3}{3}$ represented variable costs.

Table 16 Frequencies and percentages for the level of accuracy for task 9

Level of accuracy	Total frequency	%
Correct	17	37
Partially correct	17	37
Incorrect	8	17.3
No answer	4	8.7
Total	46	100

Table 17 Frequencies and percentages for the type of configuration activated in the answers to task 9

Types of configuration	Frequency by group		Total frequency	%
	1	2		
Calculation of variable costs	9	8	17	37
Calculation of total cost	7	10	17	37
Other algebraic manipulations	5	3	8	17.3
No solution provided	2	2	4	8.7
Total	23	23	46	100

The third type of configuration, 'other algebraic manipulations', was activated in answers in which the students did not give any meaning to the derivative (i.e. marginal cost function), and, in general, for the objects involved in the task, in the economic context. Thus, for example, procedures centred on the calculation of the derivative of the marginal cost function were activated in the answers, by means of the application of the proposition of derivation. In this way, we can observe once again how 34 students (correct and partially correct answers) give the antiderivative the meaning of 'inverse process of derivation'.

Task 10: Solution of Ordinary Differential Equations

The main objective of this task was to explore the process followed by the students in order to find the antiderivative, by means of a problem in which they needed to describe how they obtain the solution of a first-order differential equation. Additionally, by means of the descriptions of the students, it was also intended to explore the meaning that they give to the constant C, known as constant of integration, in order to see if they understand the 'inverse process' that finding an antiderivative implies. Table 18 shows the results for the level of accuracy of the answers provided for task 10.

Needless to say, these students had serious difficulties solving the task presented. Only five of them were able to describe, from a correct mathematical point of view, the process that they followed in order to find the solution to the differential equation presented. Twelve of them (26.1%) omitted the constant of 'integration' in their solutions, so we labelled their answers as partially correct. Sixty-three percent of the students did not answer or answered something 'incongruent' (not valid or senseless from a mathematical point of view). The main cause mentioned by this 63% of the students, either orally at the moment that the questionnaire was administered or in writing, in the box intended for the answer to the task, was that they did not remember or did not know how to solve a differential equation.

Table 18 Frequencies and percentages for the level of accuracy for task 10

Level of accuracy	Total frequency	%
Correct	5	10.9
Partially correct	12	26.1
Incorrect	12	26.1
No answer	17	36.9
Total	46	100

Table 19 Frequencies and percentages for the type of configuration activated for task 10

Types of configuration	Frequency by group		Total frequency	%
	1	2		
Verbal	3	8	11	23.9
Symbolic	7	7	14	30.4
Verbal-symbolic	2	2	4	8.7
No solution provided	11	6	17	37.0
Total	23	23	46	100

Regarding the types of cognitive configuration activated in the answers, these were of three types (Table 19) and were classified according to the type of linguistic element used in their arguments. The first, 'verbal', is a configuration that was activated in answers in which the verbal-descriptive language to narrate the procedure that they had to follow in order to solve a differential equation, but without 'developing' such procedures symbolically; in other words, there is a description of what should be done, but it is not actually performed. Only one student who activated this type of configuration gave a correct answer.

The second type of configuration, 'symbolic' was activated in answers that centred their arguments on the procedure itself of calculation of the solution; in other words, they solved the differential equation symbolically without describing with words the process they followed. The third configuration activated was a mixture of the two previous configurations. Four students (two Colombian and two Mexican) described the procedure and the properties/propositions used in the calculation of the solution, verbally. Three of the students, who mobilized the third configuration, 'verbal-symbolic', answered the task correctly.

Task 11: Application of the Antiderivative in Physics

The goal of task 11 was similar to task 10. The purpose was to explore the process followed by the students to find the function for the position of an object, given the function of velocity. In general, as shown in Table 20, the students did not have difficulties giving a correct answer of the type, '…if we have the function of position, we derive and then obtain the function of velocity, so if we calculate the antiderivative of the function of velocity, we will obtain the function of initial position'. This type of answers that we just exemplified, which centred on the verbal description of the procedures and properties/propositions, was connected to the configuration 'verbal descriptions' (Table 21). Once again, we observed how a high percentage of students

Table 20 Frequencies and percentages for the level of accuracy for task 11

Level of accuracy	Total frequency	%
Correct	35	76.1
Incorrect	5	10.9
No answer	6	13
Total	46	100

Table 21 Frequencies and percentages for the type of configuration activated in task 11

Types of solutions	Frequency by group		Total frequency	%
	1	2		
Verbal descriptions	14	15	29	63
Physics relations	4	7	11	24
No solution provided	5	1	6	13
Total	23	23	46	100

conceived the idea antiderivative as a 'direct inverse process', which is justified with the proposition that many of them (29) explicitly stated '…we will obtain the function of initial position', which indicates that they were not thinking of the family of functions that they would obtain by calculating the antiderivative (Kiat, 2005), but of the concrete element of such family from which the function of velocity was obtained.

In other types of correct answers (6), the configuration of 'relations of physics' was activated, and the task was justified through propositions of the type '…velocity is a change in the position of an object with respect to time and it is defined by the equation $\frac{ds}{dt} = v(t)$, this means that the function of position is an antiderivative of the function of velocity'. Other answers that activated this second configuration and that were labelled as incorrect, focused on the development of procedures centred on the algebraic manipulation of the physical proposition/property '$v = \frac{d}{t}$', without considering the differences between the function $v(t)$ provided in the problem formulation, and the magnitude v in the Physics formula.

So, What Is the Meaning of the Antiderivative Mobilized by Civil Engineers?

As verified with the analyses conducted for each of the tasks, although these were originally planned to activate a concrete partial meaning for the antiderivative (Fig. 1), the students who participated in this study (Mexican and Colombian) gave the *antiderivative* the meaning of inverse process of derivation. However, as reflected in the results of the last five tasks of the questionnaire, the inverse process that the students confer to the antiderivative is from a more operational (procedure) point of view than a conceptual one, that is to say, the students perceive a direct inverse process in the same ways as multiplying and dividing, or adding and subtracting equal amounts. This is, when the students applied the 'rules of integration' to obtain the antiderivative of a function (derivative function), many of them were doing it thinking that they would obtain the concrete function (i.e. the particular element of the family of antiderivative functions) from which such derivative function came from, without understanding what really happens with the inverse process.

For example, if we consider A, the set of elements of the family of functions $F(x) = g(x) + C$, where $C \in \mathbb{R}$, and we take an element of that set, like $F(x) = g(x) - 4$, to go through the process of derivation, we obtain $F'(x) = g'(x)$, which is the only element that make up set B (derivatives of the elements of set A). If, next, we take $F'(x) = g'(x)$ through the process of antiderivative, we obtain set A as a result, and not the particular element $F(x) = g(x) - 4$. In other words, the antiderivative understood as the inverse process of the derivative should imply to think of it in a more conceptual rather than procedural way because it would be a *non-direct inverse process*. This had already been studied by Kiat (2005) who points out in his study that the students who omit the constant of the 'antiderivative' *C* are not aware that a constant must be written to specify the antiderivative function within a family of functions; in other words, the students are not aware that an integral is formed by a set of antiderivatives with C as a constant that varies.

Now, the meaning that the students of our study give to the antiderivative, inverse process of derivation, is related to two types of partial meanings of the antiderivative (Gordillo & Pino-Fan, 2016): sums – differences, and fluents – fluxions. From a current point of view, the first partial meaning, 'sums – differences', is activated in the mathematical practices in which an intra-mathematical language (characteristic of Mathematics) and definitions that refer to the use of a rule of integration or method of integration are used to solve mathematical situations through algebraic procedures. The second partial meaning, 'fluents – fluxions', is activated in mathematical practices in which language and definitions that are characteristic of Physics are used to provide arguments for the changes between magnitudes (acceleration, velocity, position…) and solve situations that imply physical phenomena of variation or speed.

Thus, according to the primary mathematical objects (linguistic elements definitions, propositions, procedures and arguments) mobilized by the students in their practices developed in connection to each of the tasks of the questionnaire, and according to the description above of the two partial meanings of the antiderivative, there is evidence that the meaning that future engineers conferred to the antiderivative was the *inverse process of derivation*, in the s*ense of sums-differences*.

Final Reflections

The results obtained through the questionnaire administered show evidence that the students were more successful in solving tasks that required the activation of the antiderivative in its definition of *inverse process of derivation* in the sense of the partial meaning *sums-differences.* To a lesser extent, the inverse process was activated in the sense of the partial meaning of *fluents-fluxions*. Other partial meanings of the antiderivative such as *tangents-squarings* and *elementary functions* (Gordillo & Pino-Fan, 2016) were not activated in the answers of the students.

Now the questions would be why did the engineering students of our study activate, with difficulties, one of the four partial meanings of the antiderivative? The answer to this question leads us, on the one hand, to face one of the limitations of our study: The type of problems suggested, were they appropriate for engineers, their practices and interests? Although the questionnaire that we administered was not adapted to the context and interests of engineers, the tasks were designed to activate the different partial meanings of the antiderivative, and it aimed at exploring the understanding that students who were studying their first university courses (or at the beginning of any program related to mathematics) had of such notion (Gordillo et al., 2015). However, for the purpose of this research, the questionnaire is powerful, since as previously discussed, mathematics in engineering courses is studied at the beginning of the program and then, it is expected that students apply such mathematics to different contexts (i.e. real life situations of engineering) in upcoming courses. That is why our research is relevant, after finishing their first calculus courses, do students understand what is the object antiderivative?

On the other hand, the question brings to our mind the role of the educator of engineers. The educator of future engineers should be aware, first of all, of the diversity of partial meanings of the mathematical object under study, in our case, the antiderivative (Gordillo & Pino-Fan, 2016). By understanding the use of such partial meanings in the context in which he works, the educator would have opportunities to pose problems that mobilize such meanings and, at the same time, adjust to the real needs of the engineers in training. Thus, taking into account the complexity of mathematical objects (the antiderivative in our case) and presenting to the students a representative sample of the partial meanings in which this complexity is concretized can represent an improvement in the teaching and learning of such mathematical object.

Finally, this study showed how the cognitive configurations activated by the Mexican students were the same, and with similar frequencies of use, as the Colombian's, thus activating the same meaning of the antiderivative, already discussed above. Furthermore, if we add that the questionnaire had been administered to samples of university students, mathematicians and future teachers (Gordillo, 2015), and that the meaning mobilized by mathematicians as well as future teachers was mainly *Elementary functions*, then it seems that the meanings mobilized by our sample of engineers are not located in the context of a classroom, university or country, but it is a meaning that is shared by a *community*—as found by Bingolbali et al. (2007) in the case of the derivative—in this case the *community of civil engineering students*.

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